Asymptotic preserving schemes for hyperbolic systems of balance laws

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Several physical systems are described by evolutionary partial differential equations which have the structure of hyperbolic systems of balance laws of the form

(1) $$U_t + F(U)_x = \frac{1}{\epsilon} R(U),$$

where $U(x,t) \in \mathbb{R}^M$ and the right hand side may contain a stiff relaxation term as the parameter ϵ becomes small. Some specific examples include discrete kinetic models, shallow water with friction, extended thermodynamics.

After finite difference space discretization, one obtains a large system of ODEs. Such systems are efficiently solved by implicit-explicit schemes (IMEX) [1], which treat explicitly the non-stiff hyperbolic term, and implicitly the stiff source term [2]. If R is a relaxation, this means that i) there exists a matrix $Q \in \mathbb{R}^{m\times M}$, $m < M$, such that $QR(U) = 0\, \forall U \in \mathbb{R}^M$ and ii) the solutions of $R(U) = 0$ can be uniquely expressed as $U = E(u)$, with $u = QU \in \mathbb{R}^m$. As a result, as $\epsilon \to 0$ the system formally relaxes to a smaller system of conservation laws of the form

(2) $$u_t + f(u)_x = 0,$$

with $f = QF(E(u))$. This reduction for vanishing ϵ can be made rigorous if a suitable subcharacteristic condition is satisfied [3]. Such a property has been adopted to construct the *relaxation methods* [4]. A scheme for the numerical solution of system (1) which becomes a consistent scheme for system (2) as the relaxation parameter vanishes is said to be *Asymptotic Preserving* (AP) [5, 2].

The long time behaviour of such systems may be better described by a system containing parabolic terms (hyperbolic to parabolic relaxation). In such cases a different IMEX strategy is needed to capture the correct diffusion limit, and to avoid the parabolic CFL restriction on the time step.

IMEX schemes can be also adopted for the numerical solution of systems in which the stiffness does not appear in the additive form of Eq.(1), thus allowing the construction of high order schemes for non-linear problems which are much more cost-effective than fully implicit schemes.

For moderate values of ϵ, there might be non-trivial stationary solutions of problem (1). If one is interested in solving problems whose solution is a small perturbation of a stationary one, then it is highly advisable to adopt a *Well-Balanced* scheme (WB), i.e. a scheme which is able to maintain, within machine precision or to great accuracy, the (discrete or continuous) stationary solution, otherwise truncation error may be comparable or even greater than the small perturbation one is interested in [6].

If the source term contains a stiff relaxation and a non stiff term, i.e. for systems of the form

(3) $$U_t + F(U)_x = \frac{1}{\epsilon} R(U) + G(U,x),$$

in the limit of vanishing ϵ the system relaxes to a lower dimensional system of balance laws of the form

(4) $$u_t + f(u)_x = g(u,x),$$

where $g(u,x) = QG(E(u),x)$. In such cases nontrivial equilibria appear for all values of ϵ. Effective treatment of such problems for small to vanishing values of ϵ requires schemes which are at the same time AP and WB.

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A slightly different class of problems appears when both the hyperbolic term and the source are (equally) stiff:

$$\epsilon U_t + F(U)_x = R(U). \tag{5}$$

In such cases the system may relax to a stationary solution in a very short time. If one is interested in efficiently capturing the stationary solution, then it is advisable to adopt an implicit (or semi-implicit) scheme which is at the same time asymptotic preserving and well-balanced.

The purpose of the talk is to illustrate how to adopt IMEX of semi-implicit strategy to solve various kinds of systems containing stiff and non stiff terms, and how to combine them with modern high-order well-balanced finite volume discretization [7, 8], thus obtaining asymptotic-preserving well-balanced schemes for the classes of problems mentioned above.

Several numerical examples will be presented, which show the effectiveness of the approach.

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